

A Composition Theorem for Bisimulation Functions

Antoine Girard*

Technical Note

April 2013

Abstract

The standard engineering approach to modelling of complex systems is highly compositional. In order to be able to understand (or to control) the behavior of a complex dynamical systems, it is often desirable, if not necessary, to view this system as an interconnection of smaller interacting subsystems, each of these subsystems having its own functionalities. In this paper, we propose a compositional approach to the computation of bisimulation functions for dynamical systems. Bisimulation functions are quantitative generalizations of the classical bisimulation relations. They have been shown useful for simulation-based verification or for the computation of approximate symbolic abstractions of dynamical systems. In this technical note, we present a constructive result for the composition of bisimulation functions. For a complex dynamical system consisting of several interconnected subsystems, it allows us to compute a bisimulation function from the knowledge of a bisimulation function for each of the subsystem.

*Laboratoire Jean Kuntzmann, Université Joseph Fourier, B.P. 53, 38041 Grenoble Cedex 9, antoine.girard@imag.fr. This work has been supported by the Agence Nationale de la Recherche (VAL-AMS project - ANR-06-SETIN-018)

1 Introduction

The standard engineering approach to modelling of complex systems is highly compositional. In order to be able to understand (or to control) the behavior of a complex dynamical systems, it is often desirable, if not necessary, to view this system as an interconnection of smaller interacting subsystems, each of these subsystems having its own functionalities. System on chips, for instance, are often complex circuits that can be decomposed into smaller (and thus simpler) circuits.

Albeit the simplification of the modelling process, a modular representation of complex systems can greatly simplify the analysis process. In computer science, compositionality and concurrency [13] have been a very active research field. In the system engineering science, a compositional approach is also often used (see e.g. [10]). In this paper, we propose a compositional approach to the computation of bisimulation functions for dynamical systems.

Bisimulation functions have been introduced in [5] as a quantitative generalization of the classical notion of bisimulation relations that have been extensively and successfully used in purely discrete systems analysis [2]. Bisimulation functions measure how far two states of a system are from being bisimilar, thus enabling the quantification of the distance between trajectories originating from different states. Thus, these functions allow us to define a natural notion of neighborhood for trajectories of a system. Recently, several promising papers have shown that bisimulation functions can be used for simulation-based verification [8, 4, 11, 12] or for the computation of approximate symbolic abstractions of dynamical systems [7, 14].

In this technical note, we present a constructive result for the composition of bisimulation functions. For a complex dynamical system consisting of several interconnected subsystems, it allows us to compute a bisimulation function from the knowledge of a bisimulation function for each of the subsystem. Similar to Lyapunov functions for interconnected systems [10], a small gain condition has to be fulfilled in order to be able to compose bisimulation functions. The paper is organized as follows. First, we present the notion of interconnection of subsystems useful for compositional modelling of dynamical systems. Then, we introduce the notion of bisimulation function and develop a result on composition of bisimulation functions.

2 Compositional Modelling of Dynamical Systems

Compositional modelling allows us to see a complex dynamical system Σ as a set of several smaller subsystems $\Sigma_1, \dots, \Sigma_m$, interacting together. This is a standard engineering approach and softwares such as Simulink or Scicos gained their popularity from the possibility of modular representation of complex systems. In this section, we present the notion of interconnection of subsystems useful for compositional modelling of dynamical systems. In the following, we only define the interconnection of two subsystems; however, the extension to systems with more components is straightforward (see e.g. [15]).

Let us consider two dynamical systems, Σ_1 and Σ_2 of the following form:

$$\Sigma_i : \dot{x}_i(t) = f_i(x_i(t), u_i(t)), \quad i = 1, 2.$$

where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t) \in \mathbb{R}^{m_i}$ denote the state and input variables of Σ_i . The input vector is of the form $u_i(t) = [v_i(t), w_i(t)]$, where $v_i(t) \in \mathbb{R}^{p_i}$ denotes the inputs used for the interconnection of Σ_1 and Σ_2 and $w_i(t) \in \mathbb{R}^{q_i}$ denotes the external inputs (see Figure 1).

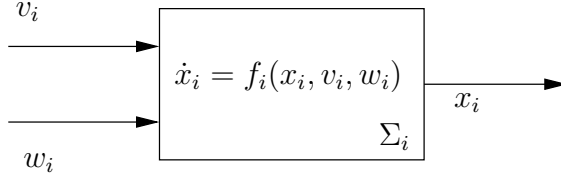


Figure 1: Subsystem Σ_i .

The interconnection of Σ_1 and Σ_2 is achieved by feeding the system inputs $v_1(t)$ and $v_2(t)$ with the state variables $x_2(t)$ and $x_1(t)$ (see Figure 2). We therefore assume that $p_1 = n_2$ and $p_2 = n_1$. Then, the interconnection of Σ_1 and Σ_2 is formally defined as follows:

Definition 1 *The interconnection of Σ_1 and Σ_2 is the dynamical system Σ given by the differential equation*

$$\Sigma : \begin{cases} \dot{x}_1(t) &= f_1(x_1(t), x_2(t), w_1(t)), \\ \dot{x}_2(t) &= f_2(x_2(t), x_1(t), w_2(t)) \end{cases}$$

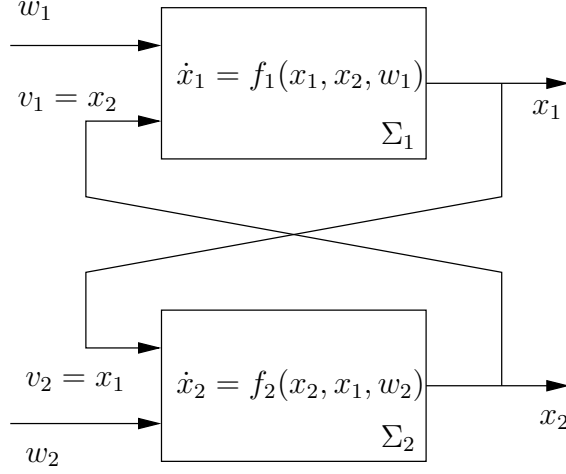


Figure 2: The composition of Σ_1 and Σ_2 .

The state of Σ is $x(t) = [x_1(t), x_2(t)] \in \mathbb{R}^n$ with $n = n_1 + n_2$ and the input of Σ is $u(t) = [w_1(t), w_2(t)] \in \mathbb{R}^m$ with $m = q_1 + q_2$. Then, the system Σ can be written under the form

$$\Sigma : \dot{x}(t) = f(x(t), u(t)),$$

that is similar to Σ_1 and Σ_2 . Then, this means that Σ can be composed with another system, enabling the hierarchical modelling of dynamical systems.

3 Composition of Bisimulation Functions

We first present the notion of bisimulation function, then we will give a result on composition of bisimulation functions.

3.1 Bisimulation functions

Let us consider a dynamical system of the form

$$\Sigma : \dot{x}(t) = f(x(t), u(t))$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Bisimulation functions have been introduced in [5] as a quantitative generalization of the classical notion of bisimulation relations that have been extensively and successfully used in purely discrete systems analysis [2].

Bisimulation functions measure how far two states of a system are from being bisimilar, thus enabling the quantification of the distance between trajectories originating from different states. Thus, these functions allow us to define a natural notion of neighborhood for trajectories of a system. The following definition slightly differs from the original definition in [5]. It is the continuous time version of the definition given in [4] which makes it suitable for simulation-based verification.

Definition 2 *A smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a bisimulation function for Σ if*

$$\|x - x'\| \leq V(x, x') \quad (1)$$

and there exists $\lambda > 0$, $\gamma \geq 0$ such that for all $x \in \mathbb{R}^n$, $x' \in \mathbb{R}^n$, $\forall u \in \mathbb{R}^m, u' \in \mathbb{R}^m$,

$$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq -\lambda V(x, x') + \gamma \|u - u'\|. \quad (2)$$

Bisimulation functions have the following property which makes them suitable tools for simulation-based verification [8, 4, 11, 12] or for the computation of approximate symbolic abstractions of dynamical systems [7, 14].

Theorem 1 *Let us consider $x(t)$ and $x'(t)$ be the trajectories of Σ given by*

$$\dot{x}(t) = f(x(t), u(t)) \text{ and } \dot{x}'(t) = f(x'(t), u'(t)).$$

Then, we have for all $t \geq 0$

$$\|x(t) - x'(t)\| \leq V(x(t), x'(t)) \leq e^{-\lambda t} V(x(0), x'(0)) + \frac{\gamma}{\lambda} \|u - u'\|_\infty$$

where $\|u - u'\|_\infty = \sup_{t \geq 0} \|u(t) - u'(t)\|$.

Proof: From equation (1), we have the first inequality. From equation (2), we have

$$\frac{dV(x(t), x'(t))}{dt} \leq -\lambda V(x(t), x'(t)) + \gamma \|u(t) - u'(t)\| \leq -\lambda V(x(t), x'(t)) + \gamma \|u - u'\|_\infty$$

Let $\eta(t) = e^{-\lambda t} V(x(0), x'(0)) + \frac{\gamma}{\lambda} \|u - u'\|_\infty$, it is a solution of the differential equation

$$\dot{\eta}(t) = -\lambda \eta(t) + \gamma \|u - u'\|_\infty.$$

Moreover, $V(x(0), x'(0)) \leq \eta(0)$; then, from the funnel theorem (see e.g. [9]), it follows that for all $t \geq 0$, $V(x(t), x'(t)) \leq \eta(t)$. ■

The practical computation of bisimulation functions is out of the scope of this technical note. However, we refer the interested reader to [3, 6] for computational methods applying to linear and nonlinear dynamical systems.

Let us remark that the previous theorem clearly shows the existing relation between the notion of bisimulation function and the notion of incremental input-to-state stability [1] (close initial states and close inputs lead to close trajectories of Σ). This connection was already pointed out in the work [14] where incremental input-to-state stability was shown sufficient for the existence of approximately bisimilar symbolic abstractions of a dynamical system.

3.2 A Composition Result for Bisimulation Functions

We now consider the problem of composing bisimulation functions. For complex systems that consists of several interconnected subsystems, it is interesting to develop compositional analysis methods. Let us assume that we are given a bisimulation function for each subsystem, then the question is whether it is possible or not to compose these functions to design a bisimulation function for the global system. The following result shows that the composition is possible under a small gain condition. It has similarities with [10] where a compositional result for ISS-Lyapunov functions is developed.

Theorem 2 *Let Σ_1 and Σ_2 be dynamical systems and let Σ be the interconnection of Σ_1 and Σ_2 as defined in Definition 1. Let V_1 and V_2 be simulation functions for Σ_1 and Σ_2 , we denote by λ_1 and γ_1 (respectively λ_2 and γ_2) the real numbers such that equation (2) holds for V_1 (respectively V_2). Then, under the small gain condition $\frac{\gamma_1\gamma_2}{\lambda_1\lambda_2} < 1$, there exists V a bisimulation function for Σ of the form:*

$$V(x, x') = \alpha_1 V_1(x_1, x'_1) + \alpha_2 V_2(x_2, x'_2) \text{ where } x = [x_1, x_2], x' = [x'_1, x'_2]. \quad (3)$$

The couple (α_1, α_2) can be chosen as follows

$$\begin{cases} \frac{\gamma_2}{\lambda_1} < \alpha_1 < \frac{\lambda_2}{\gamma_1} & \text{and} & \alpha_2 = 1 & \text{if } \lambda_1 \leq \gamma_2 \\ \alpha_1 = 1 & \text{and} & \frac{\gamma_1}{\lambda_2} < \alpha_2 < \frac{\lambda_1}{\gamma_2} & \text{if } \lambda_2 \leq \gamma_1 \\ \alpha_1 = 1 & \text{and} & \alpha_2 = 1 & \text{in the other cases.} \end{cases} \quad (4)$$

Proof: Let V be a function of the form (3), we look for conditions on α_1 and α_2 ensuring that V is a bisimulation function for Σ . First, let us remark that if $\alpha_1 \geq 1$

and $\alpha_2 \geq 1$ then,

$$V(x, x') \geq V_1(x_1, x'_1) + V_2(x_2, x'_2) \geq \|x_1 - x'_1\| + \|x_2 - x'_2\|$$

because V_1 and V_2 satisfy equation (1). Then, by remarking that

$$\|x - x'\| = \sqrt{\|x_1 - x'_1\|^2 + \|x_2 - x'_2\|^2} \leq \|x_1 - x'_1\| + \|x_2 - x'_2\|,$$

it follows that V satisfies equation (1) as well. Let $u = [w_1, w_2]$, $u' = [w'_1, w'_2]$ be inputs of Σ . Then, we have

$$\begin{aligned} \frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') &= \alpha_1 \frac{\partial V_1}{\partial x} f_1(x_1, x_2, w_1) + \alpha_2 \frac{\partial V_2}{\partial x} f_2(x_2, x_1, w_2) \\ &\quad + \alpha_1 \frac{\partial V_1}{\partial x'} f_1(x'_1, x'_2, w'_1) + \alpha_2 \frac{\partial V_2}{\partial x'} f_2(x'_2, x'_1, w'_2) \\ &\leq \alpha_1 (-\lambda_1 V_1(x_1, x'_1) + \gamma_1 \|x_2, w_1\| - \|x'_2, w'_1\|) \\ &\quad + \alpha_2 (-\lambda_2 V_2(x_2, x'_2) + \alpha_2 \gamma_2 \|x_1, w_2\| - \|x'_1, w'_2\|) \end{aligned}$$

because V_1 and V_2 satisfy equation (2). Further, we have

$$\|x_2, w_1\| - \|x'_2, w'_1\| = \sqrt{\|x_2 - x'_2\|^2 + \|w_1 - w'_1\|^2} \leq \|x_2 - x'_2\| + \|w_1 - w'_1\|$$

and

$$\|x_1, w_2\| - \|x'_1, w'_2\| = \sqrt{\|x_1 - x'_1\|^2 + \|w_2 - w'_2\|^2} \leq \|x_1 - x'_1\| + \|w_2 - w'_2\|.$$

Therefore,

$$\begin{aligned} \frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') &\leq \alpha_1 (-\lambda_1 V_1(x_1, x'_1) + \gamma_1 \|x_2 - x'_2\| + \gamma_1 \|w_1 - w'_1\|) \\ &\quad + \alpha_2 (-\lambda_2 V_2(x_2, x'_2) + \gamma_2 \|x_1 - x'_1\| + \gamma_2 \|w_2 - w'_2\|). \end{aligned}$$

Then, since V_1 and V_2 satisfy equation (1), it follows that

$$\begin{aligned} \frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') &\leq \alpha_1 (-\lambda_1 V_1(x_1, x'_1) + \gamma_1 V_2(x_2, x'_2) + \gamma_1 \|w_1 - w'_1\|) \\ &\quad + \alpha_2 (-\lambda_2 V_2(x_2, x'_2) + \gamma_2 V_1(x_1, x'_1) + \gamma_2 \|w_2 - w'_2\|) \\ &\leq -(\alpha_1 \lambda_1 - \alpha_2 \gamma_2) V_1(x_1, x'_1) + \alpha_1 \gamma_1 \|w_1 - w'_1\| \\ &\quad -(\alpha_2 \lambda_2 - \alpha_1 \gamma_1) V_2(x_2, x'_2) + \alpha_2 \gamma_2 \|w_2 - w'_2\|. \end{aligned}$$

Let us assume that $\alpha_1\lambda_1 - \alpha_2\gamma_2 > 0$ and $\alpha_2\lambda_2 - \alpha_1\gamma_1 > 0$, then let us define

$$\lambda = \min\left(\frac{\alpha_1\lambda_1 - \alpha_2\gamma_2}{\alpha_1}, \frac{\alpha_2\lambda_2 - \alpha_1\gamma_1}{\alpha_2}\right) \text{ and } \gamma = \alpha_1\gamma_1 + \alpha_2\gamma_2.$$

By remarking that $\|w_1 - w'_1\| \leq \|u - u'\|$ and $\|w_2 - w'_2\| \leq \|u - u'\|$ it follows that

$$\frac{\partial V}{\partial x}f(x, u) + \frac{\partial V}{\partial x'}f(x', u') \leq -\lambda V(x, x') + \gamma\|u - u'\|.$$

Therefore, we proved that if $\alpha_1 \geq 1$, $\alpha_2 \geq 1$, $\alpha_1\lambda_1 - \alpha_2\gamma_2 > 0$ and $\alpha_2\lambda_2 - \alpha_1\gamma_1 > 0$, then V is a bisimulation function for Σ . Let us show that a necessary and sufficient condition for the existence of a couple (α_1, α_2) satisfying these four inequalities is $\frac{\gamma_1\gamma_2}{\lambda_1\lambda_2} < 1$. Let the inequalities hold, then particularly,

$$\alpha_1\lambda_1\lambda_1 > \alpha_2\gamma_2\lambda_2 > \alpha_1\gamma_1\gamma_2.$$

It follows that necessarily $\frac{\gamma_1\gamma_2}{\lambda_1\lambda_2} < 1$. Conversely, if $\frac{\gamma_1\gamma_2}{\lambda_1\lambda_2} < 1$, there are only three possible configurations shown on Figures 3, 4 and 5. Then, by choosing α_1 and α_2 as in equation (4), the four inequalities hold. ■

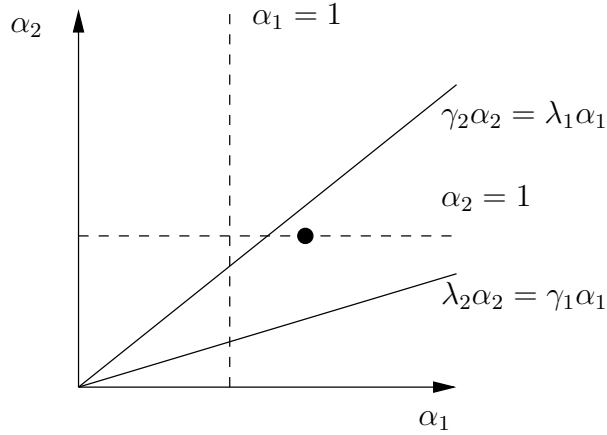


Figure 3: Configuration 1: $\lambda_1 \leq \gamma_2$.

This theorem provides us with a method to compute compositionally bisimulation functions for composite systems. Note that it is subject to a small gain condition that is $\frac{\gamma_1\gamma_2}{\lambda_1\lambda_2} < 1$. Let us remark that the choice of the couple (α_1, α_2) given in equation (4) is only one possible choice among many others satisfying the inequalities $\alpha_1 \geq 1$, $\alpha_2 \geq 1$, $\alpha_1\lambda_1 - \alpha_2\gamma_2 > 0$ and $\alpha_2\lambda_2 - \alpha_1\gamma_1 > 0$. Another choice might be more suitable depending on the considered application.

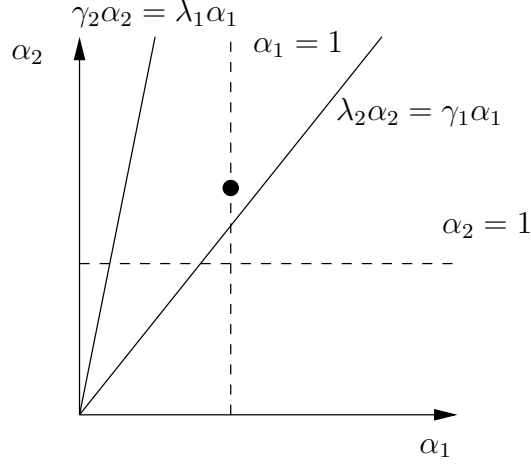


Figure 4: Configuration 2: $\lambda_2 \leq \gamma_1$.

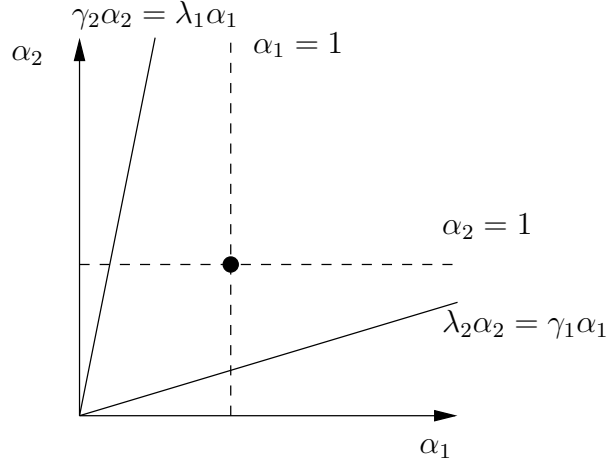


Figure 5: Configuration 3: other cases.

4 Conclusions

In this technical note, we presented a constructive result for the composition of bisimulation functions. For a complex dynamical system consisting of several interconnected subsystems, it allows us to compute a bisimulation function from the knowledge of a bisimulation function for each of the subsystem. Similar to Lyapunov functions for interconnected systems, a small gain condition has to be fulfilled in

order be able to compose bisimulation functions.

In the context of the VAL-AMS project, this result shall be useful for the computation of bisimulation functions for large scale analog circuits which can be seen as the interconnection of smaller circuits. The knowledge of a bisimulation function is required for simulation-based verification [8, 4, 11, 12] or for the computation of approximate symbolic abstractions of dynamical systems [7, 14].

References

- [1] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Trans. Automatic Control*, 47(3):410–421, March 2002.
- [2] E.M. Clarke, O. Grumberg, and D. Peled. *Model Checking*. MIT Press, 1999.
- [3] A. Girard. Reachability of uncertain linear systems using zonotopes. In *Hybrid Systems : Computation and Control*, volume 3414 of *LNCS*, pages 291–305. Springer, 2005.
- [4] A. Girard. Simulation-based techniques for verification of dynamical systems. Technical report, 2007. VAL-AMS Deliverable.
- [5] A. Girard and G. J. Pappas. Approximation metrics for discrete and continuous systems. *IEEE Transactions on Automatic Control*, 52(5):782–798, May 2007.
- [6] A. Girard and G.J. Pappas. Approximate bisimulation relations for constrained linear systems. *Automatica*, 43(8), 2007.
- [7] Antoine Girard. Approximately bisimilar finite abstractions of stable linear systems. In *Hybrid Systems: Computation and Control*, volume 4416 of *LNCS*, pages 231–244. Springer, 2007.
- [8] Antoine Girard and George J. Pappas. Verification using simulation. In *Hybrid Systems: Computation and Control*, volume 3927 of *LNCS*, pages 272–286. Springer, 2006.
- [9] J.H. Hubbard and B.H. West. *Differential equations: a dynamical systems approach*. Springer, 1995.

- [10] Z.P. Jiang, I.M.Y. Mareels, and Y. Wang. A lyapunov formulation of nonlinear small gain theorem for interconnected iss systems. *Automatica*, 32(9):1211–1215, 1996.
- [11] A. Agung Julius, Georgios E. Fainekos, Madhukar Anand, Insup Lee, and George J. Pappas. Robust test generation and coverage for hybrid systems. In *Hybrid Systems: Computation and Control*, volume 4416 of *LNCS*, pages 329–342. Springer, 2007.
- [12] F. Lerda, J. Kapinski, E.D. Clarke, and B.H. Krogh. Verification of supervisory control software using state proximity and merging. In *Hybrid Systems : Computation and Control*, LNCS. Springer, 2008. To appear.
- [13] R. Milner. *Communication and concurrency*. Prentice Hall, 1989.
- [14] G. Pola, A. Girard, and P. Tabuada. Approximately bisimilar symbolic models for nonlinear control systems. 2007. Submitted.
- [15] Y. Tazaki and J. Imura. Bisimilar finite abstractions of interconnected systems. In *Hybrid Systems : Computation and Control*, LNCS. Springer, 2008. To appear.